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AN ALGORITHM FOR THE NONLINEAR ANALYSIS OF COMPOUND BIFURCATION

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The paper presents a procedure for the localized analysis of compound bifurcations. A full range of phenomena are embraced, including loci of equilibria, secondary bifurcation, and imperfection sensitivity, the scheme showing how to generate the appropriate lowest-order non-trivial equations of interest. A number of aids to solution are presented, including the concepts of generalized imperfection and generalized loading parameter.

The scheme is developed by using a general non-diagonalized format suitable for numerical analysis, but the special diagonalized form can also be used to good effect. This is illustrated in the application to the interactive buckling of stiffened plates and shells, when local and overall buckling occur simultaneously or nearly so. The modelling relies heavily on the elimination-of-passive-coordinates routine of the general scheme. The study shows that the parabolic umbilic catastrophe is the key phenomenon for most such problems.

Finally, the branching analysis is fully illustrated for semi-symmetric branching, where one of the contributing bifurcations is symmetric and the other is asymmetric. In all, ten different loci are treated, including the full imperfection sensitivity at complete and near coincidence plotted in three-dimensional form; these relate to an earlier stiffened-plate formulation. The general scheme is thus made directly accessible for any problem that exhibits a bifurcational manifestation of either the elliptic or hyperbolic umbilic catastrophe.

Introduction

As mathematical modelling gains in sophistication, bifurcation theories are playing an increasingly important role in a growing number of scientific disciplines. However the number of different approaches to the study of bifurcation, and the class of phenomena to which the word is applied, are also growing. For an indication of the breadth of interest we refer to the proceedings of the 1977 New York Academy of Sciences Conference on Bifurcation Theory and Applications in Scientific Disciplines (Gurel & Rössler (ed.) 1979). Here we find studies in pure mathematics, hard sciences such as physics, chemistry and engineering, biology, and social sciences such as ecology and economics, and a wide range of phenomena are discussed, from the now well known elementary catastrophes of Thom (1975) to the so-called 'strange attractors' (Rössler 1979); the latter exhibit seemingly chaotic behaviour while being governed by relatively simple differential equations, and have a possible application in the modelling of turbulence (Ruelle 1979).

A bifurcation theory is usually developed under the umbrella of a specific discipline, and there is thus a wide variation in overall emphasis, depending on specialist needs and conventions; as a result some confusion has arisen over terminology, even over the meaning of 'bifurcation' itself. This is unfortunate, especially in the light of recent attempts to cross disciplinary boundaries (see, for example, Zeeman 1977, or Poston & Stewart 1978), but must be regarded as inevitable, at least for the foreseeable future. It calls for careful definitions, wherever ambiguities can arise.

Historically, structural mechanics has proved a rich source of bifurcation phenomena, largely because of inherent optimization procedures which form an integral part of any design process (Thompson & Hunt 1977*a*). Frequently these have led to 'perfect' structural designs with load-carrying properties which are unobtainable in practice, and the phenomenon of *imperfection sensitivity* results (Koiter 1945, Thompson & Hunt 1973); sometimes they call for coincidence of critical loads, leading to *compound* bifurcations (with co-rank greater than one). In this paper a comprehensive algorithm is presented for the localized analysis of these phenomena, as they arise in structural systems. The scheme is of course available for application

in other disciplines, but it is developed particularly with the problems of compound bifurcation in mind, and other fields have yet to develop a comparable interest in their analysis.

More specifically, we suppose in the first instance that, at any stage in its evolution, our system can be described by a potential function $V(Q_i)$ where Q_i represents a set of n internal variables or *generalized coordinates*. This exhibits a minimum-seeking gradient dynamic, which limits the phenomena to the elementary catastrophes (Thom 1975), and the mechanics applications to *predominately conservative* systems; by this we mean that a little positive-definite viscous damping is necessary during fast dynamic response of the system, to allow it to settle down to a stable equilibrium state in a potential well. It is important to point out that most elastic buckling failures fall into this category, and there are also indications that cases of plastic buckling with no elastic unloading can be treated from a similar quasi-elastic standpoint (Hutchinson 1974).

The state space Q_i is chosen as finite dimensional, and the analysis thus relates to a discrete rather than a continuous view of reality. We are not to concern ourselves with the considerable difficulties associated with reducing, in a mathematically rigorous way, the latter to the former; we merely note that it can often be achieved by embedding the problem in Banach or Hilbert space, and refer to Poston & Stewart (1978) for a readable modern synopsis of the situation. Nevertheless it must be stressed that both are essentially models, and it is not immediately clear whether genuinely infinite-dimensional pecularities, exclusive to a continuum, can be actually realized; a physical system, after all, comprises a large but finite number of atoms. We can say, however, that the discrete approach is the simpler, and the scheme presented here, making no appeal to any inherent diagonalization, has a clear and immediate application in finite-element modelling.

Next we allow controlled parametric changes, writing the potential function as

$$V = V(Q_i, \Lambda^j), \tag{1}$$

where Λ^j represents a set of h external control parameters; they may be genuine controls, the load on a structure for instance, or imperfections (perturbations) in the system. Interest often centres on the number of these required for *structural stability*, such that the underlying topology suffers no change of form with further perturbation. This concept was introduced by the Russians Andronov and Pontriagin in the 1930s, and is the basis of Thom's catastrophe theory, as well as a more refined classification due to Golubitsky & Schaeffer (1979*a–c*), in which the two types of control parameter are taken as unmixable; this is clearly the appropriate setting for structural mechanics (Thompson & Hunt 1977*a*).

The segregation of the controls generates fundamental differences in outlook. Separating a single special *bifurcational parameter* Λ from the rest, we find that interest naturally focuses on *equilibrium paths*, traced by varying Λ with the other Λ^j held constant (Thompson & Hunt 1977*b*), rather than the more general equilibrium surface or manifold (Thom 1975, Zeeman 1977); this evolution with Λ can be seen as taking place in a completely different time scale from that of the remainder of the Λ^j. It is the topology of the paths that Golubitsky and Schaeffer investigate for structural stability, rather than that of the V-surface itself; this often demands more control parameters than the catastrophe theory classification.

This shift of emphasis shows why confusion has arisen over the meaning of bifurcation. We take it here in the sense of Poincaré (1885), as the crossing of two or more separate equilibrium paths (of some perfect system) giving divergent responses in imperfect systems (Thompson &

Hunt 1973); some more mathematical treatments (see, for example, Rabinowitz 1977) have bifurcations in the form of the V-surface in mind. The two views do overlap, but the former specifically excludes limit points, where a single equilibrium path reaches a locally extreme value of Λ in Q_i–Λ-space, while the mathematical definition does not. There are clear computational reasons behind excluding limit points from our analytical development, and they are dealt with fully and usefully elsewhere (Thompson 1979).

The archetypal bifurcation problem of structural mechanics has one equilibrium path, single-valued with respect to Λ, with a relatively simple perhaps trivial solution, often a result of some in-built symmetry in the system. We assume that this *fundamental path* is known, and that with increasing Λ it loses its stability at an m-fold compound critical point (co-rank m). The scheme presented here sets up locally valid (asymptotically exact) equations of low order for a complete range of phenomena associated with this point of bifurcation. Ease of solution depends on the problem in hand, and we find, for the full analysis of compound branching, that numerical techniques are essential. The analytical tool is the perturbation method, relying on the implicit function theorem for rigour, and we start with the elimination of $n-m$ passive coordinates to reduce the essential degrees of freedom to m; this corresponds to the splitting lemma of mathematical texts, which we note also appears in an infinite-dimensional version. Further details can be found in Poston & Stewart (1978).

The scheme embraces all our earlier perturbation studies of bifurcation (Thompson & Hunt 1973, Hunt 1977, 1979), as well as others (Sewell 1965, Huseyin & Mandadi 1977, for example), via two conceptual generalities. First we can choose between a number of different sets of governing equations, depending on the problem in hand. Secondly and more importantly, the number, h, of Λ^j is left unspecified throughout the general development, and particular types of Λ^j are only specified as they emerge naturally from the flow of analysis. Thus the bifurcational parameter Λ, the imperfection parameters ϵ^i, the generalized imperfection ϵ, and others are defined according to the needs of the general theory. The analyst has only to adjust the governing equations and control parameters to suit his problem, and he can immediately write down the appropriate low-order localized equations. A number of aids to solution are also described.

The rationalization of our earlier work also includes a new simplified notation, a direct legacy of the general representation of the controls. We note that coefficients of Taylor expansions are left in general form, rather than being replaced by unity or some other convenient number as is common in more theoretical studies; the analysis can thus be directly and quantitatively applied to an appropriate potential function without preliminary manipulation.

The generalized approach is fully illustrated in an application to interactive buckling in stiffened structures, a difficult problem of considerable importance in both box-girder and offshore structural design. Here the elimination of passive coordinates routine, in a diagonalized form, is the principal tool, and with symmetry considerations allows a full description of the phenomenon. Quantitative evaluation of key energy-coefficients is demonstrated for a simple stiffened plate problem.

Finally, the bifurcation analysis is specifically illustrated for semi-symmetric points of bifurcation, which correspond to the elliptic and hyperbolic umbilic catastrophes, with a maximum number of four specified controls. In all, low-order equations are set up for ten different loci – including equilibrium paths of the perfect system, full imperfection-sensitivity, and secondary bifurcations – both for complete coincidence of two contributing bifurcations on the funda-

mental path and with them separated by variation of one of the controls. These are presented in tabular fashion, and methods of solution are indicated in all cases. Computer-drawn, three-dimensional imperfection-sensitivity plots are included for one typical case, a homoeoclinal point of bifurcation corresponding to the stiffened plate of Tvergaard (1973); entirely similar plots for the other semi-symmetric points, monoclinal and anticlinal, can be found in Hunt *et al.* (1979).

1. Bifurcational formalism

Consider a system governed by a gradient potential function $V(Q_i, \Lambda^j)$. We take it as axiomatic first that a stationary value of V with respect to the generalized coordinates Q_i is necessary and sufficient for equilibrium of the system, and secondly that a complete relative minimum of V with respect to the Q_i is necessary and sufficient for the stability of an equilibrium state (Thompson & Hunt 1973). Clearly these concepts imply some measure of time, but we assume that the associated response of the system is instantaneous, in the (evolution) time scale of the control parameters. We have the *n equilibrium equations* $V_i = 0$, and *critical* equilibrium is further defined by the second variation V_{ij} becoming singular. Here a subscript denotes partial differentiation with respect to the corresponding generalized coordinate (Thompson & Hunt 1973).

Next we assume that one of the controls is special, in the sense that its associated evolution takes place instantaneously in the time scale of the rest (Thompson & Hunt 1977*a*, Golubitsky & Schaeffer 1979*a*). This model has its origins in the general theory of elastic structures, where loads and imperfections are traditionally separated both theoretically and experimentally, but clearly has a wider scope for application, particularly in other branches of the physical sciences. We distinguish the special control by the lack of a superscript and designate it simply Λ, when the occasion demands that it be separately identified.

(*a*) *The fundamental path*

We ensure that the system experiences a bifurcation, as defined in the Introduction, in the following way. We identify a single equilibrium path, traced by varying Λ with the remainder of the Λ^j constant, which always, in the region of interest, has a component in the Λ-direction in Q_i–Λ-space. This specifically excludes limit points (Thompson 1979), along with some higher-order phenomena, and the path is thus single valued with respect to Λ. Most importantly from the analytical point of view, it allows a simplifying transformation to a new potential function, involving a localized set of incremental generalized coordinates. We shall refer to this path as the *primary* or *fundamental* path.

In structural mechanics formulations a fundamental path of this type is often exhibited by the *perfect* system, or that envisaged by the designer. It frequently represents some simple perhaps trivial solution, a result of underlying symmetries. *Imperfect* systems are generated at different but constant values of others of the Λ^j, the imperfection parameters. Limit points are not excluded from the paths of imperfect systems.

It is well known that, on varying a single control, there is a unique equilibrium path through a non-critical equilibrium state (Thompson & Hunt 1973). Interest naturally focuses on initially stable paths, and a basic theorem for gradient systems (Thompson 1970, Thompson & Hunt 1973), since proved by Kuiper as reported by Chillingworth (1976), shows that stability can now only be lost at an intersection with a second path, a point of bifurcation. We see later that,

under such circumstances, the potential function carries special properties with respect to the single control Λ, which we henceforth refer to as the *bifurcational parameter*.

(*b*) *Incremental coordinates and associated transformation*

Formally, the single-valued fundamental equilibrium solution is written

$$Q_i = Q_i^{\mathrm{F}}(\Lambda), \tag{2}$$

and we assume it to be known. It gives a path F in Q_i–Λ-space, which varies monotonically with Λ in the region of interest, and which intersects other as yet unknown *secondary* or *post-buckling* paths at points of bifurcation.

We introduce a localized set of n incremental coordinates q_i defined by

$$Q_i = Q_i^{\mathrm{F}}(\Lambda) + q_i, \tag{3}$$

which implies a one-to-one correspondence between the Q_i and the q_i; this is clearly only true in the absence of limit points on F. With varying Λ the origin of the new coordinate frame slides along the fundamental path, spanning the full range of interest of Q_i–Λ-space; again it is only by denying limiting behaviour that this can be ensured.

We write a new potential function, in terms of the q_i:

$$W(q_i, \Lambda^j) \equiv V[Q_i^{\mathrm{F}}(\Lambda) + q_i, \Lambda^j]. \tag{4}$$

This merely amounts to substituting equations (3) into V; any constants generated can be ignored, since we are only interested in variations of V or W, never absolute values. The manoeuvre may, however, destroy certain linearities (Thompson & Hunt 1973).

The equilibrium and stability conditions, expressed axiomatically above, pass over unchanged to the new W-function, which has the properties,

$$W_i^{\mathrm{F}} = W_i'^{\mathrm{F}} = W_i''^{\mathrm{F}} = W_i'''^{\mathrm{F}} = \ldots = 0, \tag{5}$$

where subscripts denote partial differentiation with respect to the q_i, and a prime denotes partial differentiation with respect to Λ. These arise because, in mapping from Q_i–Λ-space to q_i–Λ-space, F is moved onto the Λ-axis, and a Taylor expansion of W_i in terms of Λ must thus result in $W_i = 0$ everywhere.

The derivative W_i' is of most interest here. Its vanishing at a point of bifurcation is intimately connected with the fact that here, as opposed to a limit point, a generalized load does no first-order work as an elastic structure moves through its buckling displacement (Thompson & Hunt 1973). Λ must then act initially on a quadratic form of the generalized coordinates in a Taylor expansion of the potential function about the critical point, instead of a linear form as is the case with the limit point (Thompson & Hunt 1973).

(*c*) *Active and passive coordinates*

Stability of equilibrium states on the fundamental path is governed, in the first instance, by the quadratic form $\frac{1}{2}W_{ij}^{\mathrm{F}} q_i q_j$, and we assume that the path is initially stable, so that for low Λ W_{ij}^{F} is positive definite. With increasing Λ we shall suppose that we encounter an *m-fold compound point of bifurcation* C, where $\Lambda = \Lambda^{\mathrm{C}}$ and W_{ij}^{C} is singular and of rank $n-m$ (co-rank m). We assume that no other bifurcations arise on F in the region of interest, so problems of near

coincidence are to be treated with the introduction of specific Λ^j to draw the bifurcations together.

We next split the n generalized coordinates into two distinct subsets, the active and passive coordinates. Thus $n-m$ are selected as the *passive* coordinates and we adopt Greek suffices for their use, writing them as q_α, while the remaining m become the active coordinates and we henceforth reserve Latin subscripts for their use. The only restricting condition on this segregation is that the submatrix related to just the passive coordinates must be non-singular, and so

$$|W^{\mathrm{C}}_{\alpha\beta}| = 0. \tag{6}$$

We know from the definition of the rank of a matrix that a valid segregation can always be found, but we note that it is not necessarily unique; a different choice of passive coordinates could lead to a different but equally valid analysis. However the situation is completely clear-cut if the quadratic form is diagonalized, for then the active coordinates must be the amplitudes of the critical modes.

(*d*) *Elimination of passive coordinates*

We now demonstrate how the $n-m$ passive coordinates can be eliminated from the analysis by the systematic use of an intrinsic perturbation scheme. This general method has proved most useful and versatile in elastic stability studies (Sewell 1965; Thompson & Hunt 1973; Hunt 1977, 1979), and is the major analytical tool of this paper. The scheme for eliminating the passive coordinates serves as an introduction to the underlying philosophy.

The procedure can be described as follows. We start by assuming, in implicit parametric form, a result that we are trying to obtain. This is then substituted into the appropriate non-linear equations – those which if they could be solved would give the required result. The equations become identities – we effectively constrain the system to states which satisfy them – and can be repeatedly differentiated term by term to generate an ordered series of equations, often sequentially linear. Particular solutions to these perturbation equations can then be used to construct the assumed form as a power series.

The implicit assumption that nonlinear relations can be expressed as Taylor series raises certain mathematical questions which we shall not attempt to answer here; we merely refer to the excellent account of Poston (Poston & Stewart 1976), who discusses the point in the light of alternative underlying assumptions of analyticity and determinacy. For the localized analysis of stability phenomena in conservative structural systems, such considerations are rarely necessary.

Here, we begin by assuming that the passive coordinates can be written as a function of the active coordinates and the control parameters:

$$q_\alpha = q_\alpha(q_i, \Lambda^j). \tag{7}$$

These are then substituted into the passive equilibrium equations $W_\alpha = 0$ to give the identities

$$W_\alpha[q_i, q_\beta(q_i, \Lambda^j), \Lambda^j] \equiv 0, \tag{8}$$

which define an *activity surface* in the full q_α–q_i–Λ^j-space, containing in essence a complete description of the system response (Thompson & Hunt 1973).

The left-hand side is now just a function of q_i and Λ^j, and can be repeatedly differentiated with respect to these to give the ordered series of perturbation equations. Differentiating once we have

$$\left.\begin{aligned} W_{\alpha i} + W_{\alpha\beta} q_{\beta i} &= 0, \\ W_{\alpha\beta} q^j_\beta + W^j_\alpha &= 0, \end{aligned}\right\} \tag{9}$$

and a second time,

$$\left.\begin{aligned} W_{\alpha ij}+W_{\alpha\beta i}q_{\beta j}+W_{\alpha\beta j}q_{\beta i}+W_{\alpha\beta\gamma}q_{\beta i}q_{\gamma j}+W_{\alpha\beta}q_{\beta ij} &= 0,\\ W_{\alpha\beta i}q^j_\beta+W^j_{\alpha i}+W_{\alpha\beta\gamma}q_{\beta i}q^j_\gamma+W^j_{\alpha\beta}q_{\beta i}+W_{\alpha\beta}q^j_{\beta i} &= 0,\\ W_{\alpha\beta\gamma}q^i_\beta q^j_\gamma+W^i_{\alpha\beta}q^j_\beta+W_{\alpha\beta}q^{ij}_\beta+W^j_{\alpha\beta}q^i_\beta+W^{ij}_\alpha &= 0, \end{aligned}\right\} \tag{10}$$

etc. Here subscripts on W denote partial differentiation (of the original unconstrained form, before the substitution of equations (7)) with respect to the corresponding coordinate, as do further subscripts on q_β; similarly superscripts denote partial differentiation with respect to the corresponding Λ^j. We thus have

$$q_{\beta i}\equiv\frac{\partial q_\beta}{\partial q_i},\quad q^j_\beta\equiv\frac{\partial q_\beta}{\partial\Lambda^j},\quad q^j_{\beta i}\equiv\frac{\partial^2 q_\beta}{\partial q_i\,\partial\Lambda^j}, \tag{11}$$

etc. The tensor summation convention is employed over Greek subscripts, with summation ranging over the $n-m$ passive values.

We now assume that W is written as a Taylor series, known to as high an order as necessary, and expanded about any point F on the fundamental path. Evaluating the perturbation equations at F then gives the required q_β-derivatives. We have the two sets of first-order equations,

$$\left.\begin{aligned} W_{\alpha\beta}q_{\beta i}|^{\mathrm F} &= -W^{\mathrm F}_{\alpha i},\\ W_{\alpha\beta}q^j_\beta|^{\mathrm F} &= -W^{j\mathrm F}_{\alpha}, \end{aligned}\right\} \tag{12}$$

which are linear, and non-singular by equation (6), and can be readily inverted to give $q^{\mathrm F}_{\beta i}$ and $q^{j\mathrm F}_\beta$. The equations are in standard form, so each of these derivatives can be written directly as the ratio of two determinants, the denominator being the complete determinant of $W^{\mathrm F}_{\alpha\gamma}$ and the numerator being the determinant of this same matrix with the particular $W^{\mathrm F}_{\alpha\beta}$-elements replaced by $-W^{\mathrm F}_{\alpha i}$ (or $-W^{j\mathrm F}_{\alpha}$). We see that this process comprises a total of $(m+h)(n-m)$ such calculations and may thus require the aid of a digital computer, to which it would be well suited.

We remember that the bifurcational parameter Λ is among the Λ^j, and has the properties of equation (5), so we have the particular results from the second set of equations (12):

$$q'^{\mathrm F}_\beta=0. \tag{13}$$

We note that the prime is again used, replacing one of the superscripts, when differentiation is specifically with respect to Λ.

The second-order equations can now be written in the form

$$\left.\begin{aligned} W_{\alpha\beta}q_{\beta ij}|^{\mathrm F} &= -(W_{\alpha ij}+W_{\alpha\beta i}q_{\beta j}+W_{\alpha\beta j}q_{\beta i}+W_{\alpha\beta\gamma}q_{\beta i}q_{\gamma j})|^{\mathrm F},\\ W_{\alpha\beta}q^j_{\beta i}|^{\mathrm F} &= -(W^j_{\alpha i}+W_{\alpha\beta i}q^j_\beta+W^j_{\alpha\beta}q_{\beta i}+W_{\alpha\beta\gamma}q_{\beta i}q^j_\gamma)|^{\mathrm F},\\ W_{\alpha\beta}q^{ij}_\beta|^{\mathrm F} &= -(W^{ij}_\alpha+W^i_{\alpha\beta}q^j_\beta+W^j_{\alpha\beta}q^i_\beta+W_{\alpha\beta\gamma}q^i_\beta q^j_\gamma)|^{\mathrm F}. \end{aligned}\right\} \tag{14}$$

We see that the right-hand side of each of these is known once the first-order derivatives are obtained, and the equations can thus be readily solved by the same method as above for the derivatives $q^{\mathrm F}_{\beta ij}$, $q^{j\mathrm F}_{\beta i}$, and $q^{ij\mathrm F}_\beta$. The scheme can then be continued to the next-order equations if necessary, and continued in sequentially linear fashion. One general result, which can be readily proved by induction, is that

$$q'^{\mathrm F}_\beta=q''^{\mathrm F}_\beta=q'''^{\mathrm F}_\beta=\ldots=0, \tag{15}$$

which arises from the special properties of the bifurcational parameter Λ.

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We can now write Taylor expansions of $q_\alpha(q_i, \Lambda^j)$ to as high an order as we please, and this leads to a new potential function with only m degrees of freedom, defined by the identity

$$\mathscr{W}(q_i, \Lambda^j) \equiv W[q_i, q_\alpha(q_i, \Lambda^j), \Lambda^j]. \tag{16}$$

We obtain derivatives of the new function by direct differentiation,

$$\left.\begin{aligned} \mathscr{W}_i &= W_i + W_\alpha q_{\alpha i}, \\ \mathscr{W}_{ij} &= W_{ij} + W_{\alpha i} q_{\alpha j} + W_{\alpha j} q_{\alpha i} + W_{\alpha\beta} q_{\alpha i} q_{\beta j} + W_\alpha q_{\alpha ij}, \\ \mathscr{W}_i^j &= W_i^j + W_{\alpha i} q_\alpha^j + W_\alpha^j q_{\alpha i} + W_{\alpha\beta} q_{\alpha i} q_\beta^j + W_\alpha q_{\alpha i}^j \end{aligned}\right\} \tag{17}$$

etc. and evaluation on the fundamental path now gives, after a little algebra, the results of interest:

$$\left.\begin{aligned} \mathscr{W}_i^{\mathrm{F}} &= 0, \\ \mathscr{W}_{ij}^{\mathrm{F}} &= W_{ij} + W_{\alpha i} q_{\alpha j}|^{\mathrm{F}}, \\ \mathscr{W}_i^{j\mathrm{F}} &= W_i^j + W_{\alpha i} q_\alpha^j|^{\mathrm{F}}, \\ \mathscr{W}_{ijk}^{\mathrm{F}} &= W_{ijk} + W_{\alpha jk} q_{\alpha i} + W_{\alpha ik} q_{\alpha j} + W_{\alpha ij} q_{\alpha k} + W_{\alpha\beta i} q_{\alpha j} q_{\beta k} \\ &\quad + W_{\alpha\beta j} q_{\alpha i} q_{\beta k} + W_{\alpha\beta k} q_{\alpha i} q_{\beta j} + W_{\alpha\beta\gamma} q_{\alpha i} q_{\beta j} q_{\gamma k}|^{\mathrm{F}}, \\ \mathscr{W}_{ij}^{k\mathrm{F}} &= W_{ij}^k + W_{\alpha j}^k q_{\alpha i} + W_{\alpha i}^k q_{\alpha j} + W_{\alpha ij} q_\alpha^k + W_{\alpha\beta i} q_{\alpha j} q_\beta^k + W_{\alpha\beta j} q_{\alpha i} q_\beta^k \\ &\quad + W_{\alpha\beta}^k q_{\alpha i} q_{\beta j} + W_{\alpha\beta\gamma} q_{\alpha i} q_{\beta j} q_\gamma^k|^{\mathrm{F}}, \\ \mathscr{W}_{ijkl}^{\mathrm{F}} &= W_{ijkl} + W_{\alpha jkl} q_{\alpha i} + W_{\alpha ikl} q_{\alpha j} + W_{\alpha ijl} q_{\alpha k} + W_{\alpha ijk} q_{\alpha l} \\ &\quad + W_{\alpha\beta kl} q_{\alpha i} q_{\beta j} + W_{\alpha\beta jl} q_{\alpha i} q_{\beta k} + W_{\alpha\beta jk} q_{\alpha i} q_{\beta l} + W_{\alpha\beta il} q_{\alpha j} q_{\beta k} \\ &\quad + W_{\alpha\beta ik} q_{\alpha j} q_{\beta l} + W_{\alpha\beta ij} q_{\alpha k} q_{\beta l} + W_{\alpha\beta\gamma i} q_{\alpha j} q_{\beta k} q_{\gamma l} + W_{\alpha\beta\gamma j} q_{\alpha i} q_{\beta k} q_{\gamma l} \\ &\quad + W_{\alpha\beta\gamma k} q_{\alpha i} q_{\beta j} q_{\gamma l} + W_{\alpha\beta\gamma l} q_{\alpha i} q_{\beta j} q_{\gamma k} + W_{\alpha\beta\gamma\delta} q_{\alpha i} q_{\beta j} q_{\gamma k} q_{\delta l} \\ &\quad - W_{\alpha\beta} q_{\alpha ij} q_{\beta kl} - W_{\alpha\beta} q_{\alpha ik} q_{\beta jl} - W_{\alpha\beta} q_{\alpha il} q_{\beta jk}|^{\mathrm{F}} \end{aligned}\right\} \tag{18}$$

etc., which can be used to construct a Taylor expansion of $\mathscr{W}$. The equilibrium and stability conditions (expressed by the earlier axioms) pass over unchanged to the new function (Thompson & Hunt 1973), so from now on we can operate exclusively and with confidence in terms of $\mathscr{W}$. We have in particular the special results

$$\mathscr{W}_i^{\mathrm{F}} = \mathscr{W}_i'^{\mathrm{F}} = \mathscr{W}_i''^{\mathrm{F}} = \ldots = 0, \tag{19}$$

$$\mathscr{W}_{ij}'^{\mathrm{F}} = W'_{ij} + W'_{\alpha i} q_{\alpha j} + W'_{\alpha j} q_{\alpha i} + W'_{\alpha\beta} q_{\alpha i} q_{\beta j}|^{\mathrm{F}},$$

from the properties of the bifurcational parameter Λ, and at the m-fold critical point C itself,

$$\mathscr{W}_{ij}^{\mathrm{C}} = 0, \tag{20}$$

so that $\mathscr{W}_{ij}^{\mathrm{C}}$ is null.

This completes the general perturbation method for the elimination of $n-m$ passive coordinates at an m-fold point of bifurcation, which as we discussed earlier, corresponds to the splitting lemma of mathematical texts; rigorous justification of the process lies in the implicit function theorem (Poston & Stewart 1978). The scheme makes no resort to diagonalization of potential function, and thus is particularly suited to numerical computer analysis, arising perhaps from a finite-element formulation. We note that, although we are now able to cast a problem in an m-dimensional form, the passive coordinates are not simply neglected, and any contaminating effect that they may have on the buckling modes is automatically taken into account.

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2. Bifurcation analysis

We present in this section a comprehensive perturbation approach to the analysis of m-fold bifurcation. The general treatment allows us to set up the appropriate first-order equations over a very broad spectrum of problems, including post-buckling, imperfection sensitivity, loci of secondary bifurcations, and even the locus of an m-fold point itself when necessary. Complete solution to the equations, to give asymptotically exact results for the problem in hand, is often difficult, and we demonstrate two particular aids to solution in the twin concepts of *generalized imperfection* and *generalized load*.

Having successfully eliminated the passive coordinates we shall operate exclusively with the m-degrees-of-freedom potential function,

$$\mathscr{W} = \mathscr{W}(q_i, \Lambda^j), \tag{21}$$

with the properties described at the end of the previous section. The set of control parameters Λ^j of course includes the bifurcational parameter Λ, but the composition of the remainder depends crucially on the problem in hand; thus, for example, m imperfection parameters can be identified among the Λ^j, but these are set to zero to investigate the equilibrium behaviour of a perfect system.

The governing equations to be used depend precisely on the information required. The full set of equilibrium equations $\mathscr{W}_i = 0$ will always appear. A general point of critical equilibrium is further defined by the local eigenvalue equation $\mathscr{W}_{ij}x_j = 0$, where x_j denotes a local eigenvector (Hunt 1977); this takes the specific form $\mathscr{W}_{ij} = 0$ if the locus of an m-fold critical point is sought (Hunt 1979). Secondary bifurcations are pin-pointed from among all critical states by the additional equation $\mathscr{W}_i' x_i = 0$ (Hunt 1977). A perturbation approach with a single independent variable is to be used, so all variables, including the local eigenvector x_k, are written in the parametric form

$$q_i = q_i(s),\ \Lambda^j = \Lambda^j(s),\ x_k = x_k(s), \tag{22}$$

where s is the single *perturbation parameter*, as yet undefined; it may be left in this general form or specified further, according to the demands of the analysis. Perturbation equations are to be derived initially in the most general context, but in any specific instance certain terms may not be present, as we shall see later in the analysis of semi-symmetric branching.

(a) *Equilibrium equations*

Substituting the above parametric forms into the equilibrium equations $\mathscr{W}_i = 0$ we obtain the *equilibrium identity*,

$$\mathscr{W}_i[q_j(s), \Lambda^k(s)] \equiv 0, \tag{23}$$

which is differentiated once with respect to s to give,

$$\mathscr{W}_{ij}q_j^{(1)} + \mathscr{W}_i^j \Lambda^{j(1)} = 0. \tag{24}$$

Here a superscript in parentheses denotes the number of full differentiations with respect to the perturbation parameter s, and the tensor summation convention is used, repeated subscripts denoting summation from 1 to m, and repeated superscripts denoting summation from 1 to h.

On evaluation at the m-fold bifurcation point C the first term vanishes by the criticality condition (20) and we have

$$\mathscr{W}_i^j \Lambda^{j(1)}|^{\mathrm{C}} = 0, \tag{25}$$

which at first glance seems a set of m equations in the h unknowns $\Lambda^{j(1)\mathrm{C}}$. But some of this set of derivatives make no appearance in the equations, since they are multiplied by zero coefficients; the bifurcational parameter derivatives $\Lambda^{(1)\mathrm{C}}$ would be absent, from condition (19), as would the corresponding derivative of the splitting parameter σ, of the later semi-symmetric analysis.

The control parameters that are represented are those which act on a linear form of the active coordinates, in the Taylor expansion of $\mathscr{W}$ about C. These are referred to as the *imperfection parameters*, henceforth denoted ϵ^j, and we shall suppose them to be m in number; if this is too many for a specific problem we can later set some to zero, while a formulation with more than m can always be reduced to this number by lumping similar contributions together. This reduces equations (25) to m equations in m unknowns,

$$\mathscr{W}_i^j \epsilon^{j(1)}|^{\mathrm{C}} = 0, \tag{26}$$

and assuming that the imperfections arise in the system in some typical fashion so as to render these equations non-singular and $|\mathscr{W}_i^{j\mathrm{C}}| \neq 0$, we have the important result

$$\epsilon^{j(1)\mathrm{C}} = 0. \tag{27}$$

The m imperfection parameters will be assumed, for the sake of simplicity, to be the first of the Λ^j, with j ranging from 1 to m. The remaining $h-m$ control parameters, which must comprise a non-empty set since it includes the bifurcational parameter Λ, are formally represented by the condition

$$\mathscr{W}_i^{j\mathrm{C}} = 0, \tag{28}$$

for all i and j ranging from $m+1$ to h.

The segregation of the controls into two groups is of deeper significance than at first might be supposed. We shall find in every perturbation equation that we derive, that the two remain completely separate. They thus clearly have quite different roles to play. The point is well illustrated in the next equilibrium perturbation equation.

Differentiating the equilibrium identity a second time and evaluating at the point C we obtain

$$\mathscr{W}_{ijk} q_j^{(1)} q_k^{(1)} + 2\mathscr{W}_{ij}^k q_j^{(1)} \Lambda^{k(1)} + \mathscr{W}_i^j \epsilon^{j(2)}|^{\mathrm{C}} = 0, \tag{29}$$

the first non-trivial equation of interest. We see that the summation implied by the repeated superscript of the second term ranges just over $k = m+1$ to h, by virtue of result (27), while the third term contains merely ϵ^j-derivatives from conditions (28). We have here m nonlinear equations in $m+h$ unknowns, and general solution is clearly out of the question; however, they are tractable in certain instances, as we shall see later. The process can be continued to higher order by further differentiation and evaluation, but this is not done explicitly here.

(*b*) *Critical state equations*

In imperfection-sensitivity studies it is necessary to pin-point states of critical equilibrium and this can be done via a local linear eigenvalue equation $\mathscr{W}_{ij} x_j = 0$, x_j representing the local eigenvector (Hunt 1977). Substituting the parametric forms (22) we thus obtain the *critical-state identity*:

$$\mathscr{W}_{ij}[q_k(s)\, \Lambda^l(s)]\, x_j(s) \equiv 0. \tag{30}$$

Differentiation with respect to s and evaluation at C then gives the equations of interest,

$$\mathscr{W}_{ijk}x_j q_k^{(1)} + \mathscr{W}_{ij}^{k} x_j \Lambda^{k(1)}|^{\mathrm{C}} = 0, \tag{31}$$

in general a set of m equations in $m+h$ unknowns. We see that ϵ^j-derivatives make no appearance here, and the superscript summation of the second term is again over the range $k = m+1$ to h. The process can be continued by further differentiation and evaluation, but this is not given explicitly.

In some studies it may be desirable to trace the locus of an m-fold critical point (Hunt 1979), and we can do this by replacing the above equation with its special form for m-fold criticality, $\mathscr{W}_{ij} = 0$. Substituting the parametric forms gives the *m-fold critical state identity*

$$\mathscr{W}_{ij}[q_k(s), \Lambda^l(s)] \equiv 0, \tag{32}$$

differentiation and evaluation giving m^2 equations in h unknowns,

$$\mathscr{W}_{ijk} q_k^{(1)} + \mathscr{W}_{ij}^{k} \Lambda^{k(1)}|^{\mathrm{C}} = 0. \tag{33}$$

We note that duplicates will arise among this set of equations, from the symmetry properties of the $\mathscr{W}$-derivatives. Again the ϵ^j-derivatives make no appearance, superscript summation is over $k = m+1$ to h, and the scheme could be continued if necessary.

(c) *Secondary bifurcation equations*

Finally, we may wish specifically to pin-point secondary bifurcations, from among all the critical states that can arise. We do this with the introduction, along with equilibrium and critical state equations, of the additional equation $\mathscr{W}_i' x_i = 0$, x_i denoting the critical local eigenvector as before. This states algebraically that a generalized load does no first-order work, as a system moves through its buckling displacement, and can be rigorously justified with the introduction of a set of incremental coordinates that remain fixed at the critical point of interest (Thompson & Hunt 1973). We note that the bifurcational parameter Λ must now play a key role, and the prime appears in the above equation, since the difference between a bifurcation and a limit point depends on the orientation of the Λ-direction in control space (Thompson & Hunt 1977*a*).

Substituting the parametric forms (22) into this equation we obtain the *secondary bifurcation identity*

$$\mathscr{W}_i'[q_j(s), \Lambda^k(s)]\, x_i(s) \equiv 0, \tag{34}$$

differentiation with respect to s and evaluation at C now giving the single equation in $m+h-1$ unknowns:

$$\mathscr{W}_{ij}' x_i q_j^{(1)} + \mathscr{W}_i'^{j} x_i \Lambda^{j(1)}|^{\mathrm{C}} = 0. \tag{35}$$

The second term must be included in this general formulation. However ϵ^j-derivatives will be absent as before, derivatives of Λ likewise by condition (19), and similar conditions for the remaining Λ^j-derivatives may be such that the complete term vanishes; this is certainly the case for the illustrated semi-symmetric branching analysis. As before we note that the scheme can be continued as necessary.

(d) *The generalized imperfection*

Having set up the appropriate perturbation equations we now turn our attention to their solution. For a distinct critical point with $m = 1$ this can usually be simply done, but with

$m \neq 1$ the situation becomes much more complicated. However, the analysis frequently can be simplified by systematically taking sections through the control space, thereby reducing the number of unknowns in a typical set of equations but solving them repeatedly. We now introduce one way in which this can be done, with the concept of a *generalized imperfection* (Hunt 1977). Here it is assumed that the full imperfection space ϵ^i is most conveniently scanned with a sweep of an *imperfection ray* R through the full space, use being made of a polar rather than a cartesian representation of the imperfections.

Let us therefore consider in isolation this m-dimensional imperfection space ϵ^i. We introduce a polar coordinate transformation on the imperfection parameters thus;

$$\epsilon^i = \epsilon^i(\theta_\alpha, \epsilon) = G^i(\theta_\alpha)\,\epsilon, \tag{36}$$

where α ranges from 1 to $m-1$. Here the $G^i(\theta_\alpha)$ are trigonometric functions only; for $m = 2$ they are $G^1 = \sin\theta$, $G^2 = \cos\theta$, while for $m = 3$, $G^1 = \sin\theta_1$, $G^2 = \sin\theta_2 \cos\theta_1$, and $G^3 = \cos\theta_2 \cos\theta_1$, suffice. The parameter ϵ, which clearly represents progress along a ray emerging from the origin $\epsilon^i = 0$, is termed the *generalized imperfection*.

Writing now the potential function as $\mathscr{W}(q_i, \Lambda^j, \epsilon^k)$, so that *the set Λ^j is from now on understood specifically to exclude the m imperfection parameters*, we can determine the energy level just on the ray R, where $\theta_\alpha = \theta_\alpha^{\mathrm{R}}$, by the transformation

$$\mathscr{W}(q_i, \Lambda^j, \epsilon) \equiv \mathscr{W}[q_i, \Lambda^j, G^k(\theta_\alpha^{\mathrm{R}})\,\epsilon]. \tag{37}$$

The derived perturbation equations still hold good for the new potential function of the left-hand side, which we continue to call $\mathscr{W}$, but the number of unknowns is reduced by $m-1$. Required derivatives can be found by successive differentiation and evaluation at C; those with respect to just q_i and Λ^j remain unchanged, while a dot will be used to denote partial differentiation with respect to the generalized imperfection ϵ. Thus,

$$\dot{\mathscr{W}}_i^{\mathrm{C}} = \mathscr{W}_i^{j\mathrm{C}} G^j(\theta_\alpha^{\mathrm{R}}) \tag{38}$$

etc., the repeated superscript implying summation over the range $j = 1$ to m. We see that the use of the same symbol, $\mathscr{W}$, on both sides of the transformation identity need create no confusion, since the syntax will always make it perfectly clear which representation applies. We note finally that the sweep of the ray through the full imperfection space implies considerable repetition, so the use of a computer becomes almost essential in problems of compound bifurcation.

(*e*) *The generalized loading parameter*

The generalized imperfection construction is only appropriate when the m imperfection parameters all enter the potential function in the same way, acting on a linear form of the q_i. We now concentrate on the set of Λ^j that act on quadratic terms, which we can suppose includes the bifurcational parameter Λ. When more than one such parameter is to be considered, it is sometimes useful to adopt a similar construction, thereby reducing the number to one but generating a full series of like problems as before.

In a bifurcational problem of structural mechanics, this subset of Λ^j can comprise just structural loading parameters, and so the analysis embraces the multiple loading situation. The single key parameter of the construction is thus referred to as the *generalized loading parameter*, although it need not have any such physical significance in a specific problem. The set

may include controlled geometric changes which act as *splitting parameters* to separate contributing bifurcations on the fundamental path (Hunt *et al.* 1979), or possibly other perturbations which might enter the potential function in this special way. The formulation thus corresponds closely to that of Huseyin (1975).

Let us suppose that we have r such parameters λ^j, where $r \leqslant h-m$, defined by

$$\lambda^j = \Lambda^j - \Lambda^{j\mathrm{C}}, \tag{39}$$

j scanning over the values $m+1$ to $m+r$. The λ^j are thus defined as incremental measures of the controls away from the critical point C. We see that no attempt has been made to relate the number r to m, unlike the generalized imperfection analysis.

We introduce the polar transformation

$$\lambda^i = \lambda^i(\phi_\alpha, \lambda^*) = H^i(\phi_\alpha)\lambda^*, \tag{40}$$

where α ranges from 1 to $r-1$. As with the imperfections, the $H^i(\phi_\alpha)$ are trigonometric functions only, so for $r = 2$ they are simply $H^{m+1} = \sin\phi$, $H^{m+2} = \cos\phi$ etc. The new parameter λ^* represents progress along a ray R emerging from the origin in 'load' space λ^i and is termed the *generalized loading parameter*.

Writing now the potential function as $\mathscr{W}(q_i, \Lambda^j, \epsilon, \lambda^k)$, so that Λ^j is here understood to exclude both the m imperfections and the r parameters λ^k and may well be an empty set, we can determine the energy level along the 'load' ray R, where $\phi_\alpha = \phi_\alpha^{\mathrm{R}}$, by the transformation

$$\mathscr{W}(q_i, \Lambda^j, \epsilon, \lambda^*) \equiv \mathscr{W}[q_i, \Lambda^j, \epsilon, H^k(\phi_\alpha^{\mathrm{R}})\lambda^*]. \tag{41}$$

Again the perturbation equations can be directly written down for the new potential function, which we still call $\mathscr{W}$, but the number of unknowns is reduced by $r-1$. The required derivatives can be found by successive differentiation and evaluation at C, derivatives with respect to just q_i, Λ^j or ϵ remaining unchanged. An asterisk will be used to denote differentiation with respect to λ^*, so we have

$$\mathscr{W}_i^{*\mathrm{C}} = \mathscr{W}_i^{j\mathrm{C}} H^j(\phi_\alpha^{\mathrm{R}}) = 0, \tag{42}$$

from conditions (28), and

$$\mathscr{W}_{ij}^{*\mathrm{C}} = \mathscr{W}_{ij}^{k\mathrm{C}} H^k(\phi_\alpha^{\mathrm{R}}), \tag{43}$$

the repeated superscript here of course implying summation merely over the range $k = m+1$ to $m+r$. Again the use of the same symbol, $\mathscr{W}$, on both sides of the transformation identity need create no confusion, since the syntax makes it clear which representation applies.

Care must be taken with application of the generalized loading parameter construction, however, since in some instances it could lead to solutions being missed. The parabolic umbilic catastrophe, for example, requires $r = 2$ for structural stability (Thom 1975). The imperfection-sensitivity analysis of its bifurcational manifestation, the paraclinal point of bifurcation, generates two quite different solution sets, one with $\lambda^{i(1)\mathrm{C}} \neq 0$ for both i, but the other with $\lambda^{2(1)\mathrm{C}} \neq 0$, $\lambda^{1(1)\mathrm{C}} = 0$; a polar representation of the λ^i would give no quantitative information on this second set of solutions, although a more subtle parabolic transformation could possibly be substituted. It is in such complex situations that a knowledge of the unique topology of individual catastrophes becomes invaluable.

We also note that when one of the λ^i is a splitting parameter, as in the following analysis of semi-symmetric branching, the imperfection sensitivity is most usefully shown with this parameter held constant. To draw such plots a numerical search procedure can be used. This is described during the analysis.

(*f*) *Further aids to solution*

The unknowns of a specific problem can be further reduced, both by specifying the perturbation parameter s, and by normalizing the critical local eigenfunction x_j (when appropriate). However, both manoeuvres can lead to solutions being missed, unless they are sought as special cases. We elaborate on these points in the analysis of semi-symmetric branching.

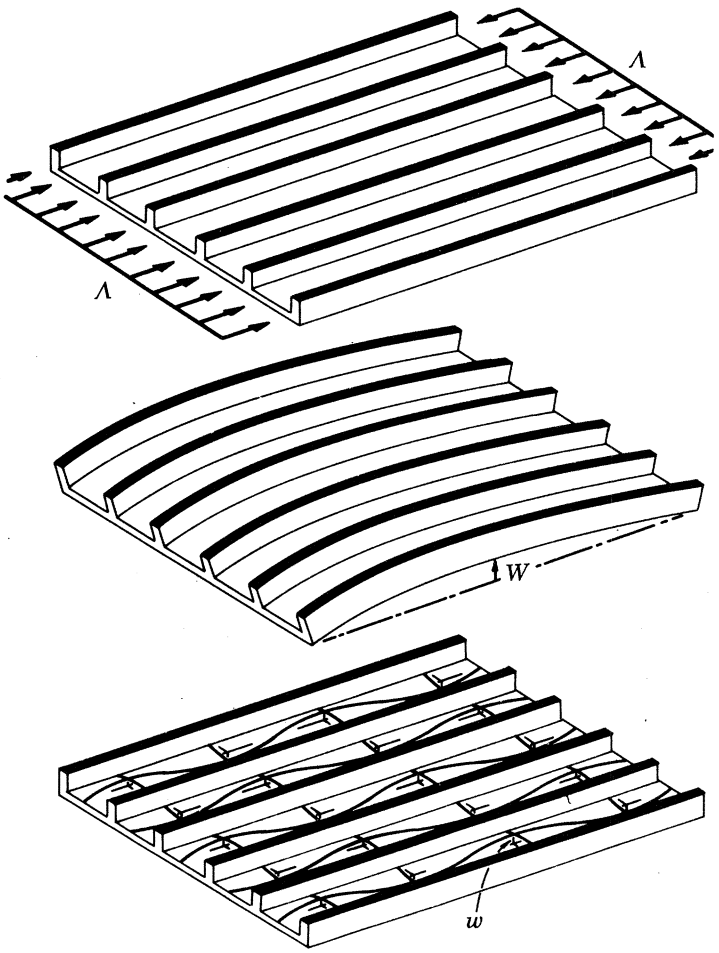

FIGURE 1. A typical stiffened plate, showing overall and local buckling modes.

3. STIFFENED STRUCTURES

Discretely stiffened components, notably plates and cylinders, are increasingly used in modern structural design for their excellent load-carrying properties. Clearly one important aspect of their behaviour must be the response under axial load. Here an optimum design will frequently call for coincidence of two critical loads, one for an overall and the other for a local mode of buckling; typical such modes for a stiffened plate are shown in figure 1. The situation is further complicated by the fact that the local mode eigenvalue is just one of a cluster of eigenvalues, all of which can significantly affect the interaction. Important theoretical advances in this

difficult area have come from Tvergaard (1973), van der Neut (1976), and Koiter & Pignataro (1976).

Here we examine the nature of the interaction in the light of the bifurcational formalism and symmetry considerations. We find that the elimination of passive coordinates provides us with a powerful tool for evaluating the contaminating effect of higher modes. Three alternative potential function models are presented to describe the dominant two-fold interaction; the complete analysis of one of these, the semi-symmetric point of bifurcation, is given in the following section. The present approach owes much to the insight of Koiter (1976), whose elegant contribution I am pleased to acknowledge.

We shall, in the main, use the plate system of figure 1 to illustrate general points. However, these can be suitably reinterpreted for any 'simple' stiffened structure in the sense of Koiter, with the modes sinusoidal in at least one direction, and

$$l^2 \ll L^2, \tag{44}$$

and L being the characteristic wavelengths for the local and overall modes respectively.

(a) *Diagonalization of potential energy*

We assume in the following that all contributing bifurcations of the perfect system arise on a fundamental path describing a uniformly compressed state, and the V-to-W transformation is known. Furthermore we suppose that the generalized coordinates directly measure critical mode amplitudes, and write them as u_i instead of q_i in recognition of this. The well known orthogonal properties of buckling modes then render the potential function diagonalized, and we shall write this as A (or $\mathscr{A}$) instead of W (or $\mathscr{W}$) to fall into line with an earlier notation (Thompson & Hunt 1973).

We suppose that the diagonalization extends to all quadratic forms of the u_i, so that A has the properties

$$A_{ij}^{\mathrm{F}} = A_{ij}^{k\mathrm{F}} = A_{ij}^{kl\mathrm{F}} = \ldots = 0 \quad \text{for} \quad i \neq j, \tag{45}$$

subscripts extending over the full (active and passive) range of the u_i, and superscripts over the full range of Λ^j. The elimination-of-passive-coordinates routine then gives the specialized results for diagonalized systems:

$$\left.\begin{aligned} u_{\alpha i}^{\mathrm{F}} &= 0, & u_{\alpha ij}^{\mathrm{F}} &= -\frac{A_{\alpha ij}}{A_{\alpha\alpha}}\Big|^{\mathrm{F}}, \\ \mathscr{A}_{ii}^{\mathrm{F}} &= A_{ii}^{\mathrm{F}}, & \mathscr{A}_{i}^{j\mathrm{F}} &= A_{i}^{j\mathrm{F}}, \\ \mathscr{A}_{ijk}^{\mathrm{F}} &= A_{ijk}^{\mathrm{F}}, & \mathscr{A}_{ii}^{\prime\mathrm{F}} &= A_{ii}^{\prime\mathrm{F}}, \\ \mathscr{A}_{ijkl}^{\mathrm{F}} &= A_{ijkl} - \sum_{\alpha=m+1}^{n} \frac{1}{A_{\alpha\alpha}} (A_{\alpha ij}A_{\alpha kl} + A_{\alpha ik}A_{\alpha jl} + A_{\alpha il}A_{\alpha jk})\Big|^{\mathrm{F}}, \end{aligned}\right\} \tag{46}$$

from equations (12), (14) and (18), where the tensor summation convention is temporarily suspended.

We see that the contaminating effect of the passive coordinates is now confined to the quartic term $\mathscr{A}_{ijkl}^{\mathrm{F}}$, and is dependent upon the cubic coefficients $A_{\alpha ij}^{\mathrm{F}}$ etc., as well as the *stability coefficients* $A_{\alpha\alpha}^{\mathrm{F}}$. Ignoring the summation, we obtain the same potential function as an m-degree-of-freedom Rayleigh–Ritz model of the system, in which the passive contribution is suppressed; re-introduction of the summation usually has a destabilizing effect, as we shall see.

(*b*) *Local modes*

Let us suppose, for the moment, that the system is restrained against overall buckling, and against all but one local mode of buckling, as shown in the lower diagram of figure 1. The wave profile is taken to be sinusoidal in the longitudinal direction, and is unspecified in the transverse direction. However we do insist that the out-of-plane deformation is zero at the stiffeners, and the transverse waveform does not vary along the length.

Clearly, for the plate of figure 1, the potential function is symmetric in the mode, in the sense that equal and opposite values of amplitude u_1, being essentially mirror images of each other about the mid-plane, give identical energy levels. For a curved panel this no longer holds. However, symmetry is frequently guaranteed by reflexion about the mid-length (even number of half-waves longitudinally), or by opposite deflexion pattern in adjacent panels, and we can as a last resort follow Koiter (1976) in taking it as a valid approximation in general, in view of the wavelength condition (44).

We must thus have a symmetric point of bifurcation C on the fundamental path, and can write

$$\mathscr{A} = \tfrac{1}{24}\mathscr{A}^{\mathrm{C}}_{1111}u_1^4 + \tfrac{1}{2}\mathscr{A}'^{\mathrm{C}}_{11}(\Lambda - \Lambda^{\mathrm{C}})u_1^2 + \text{higher-order terms}, \tag{47}$$

for the perfect system, where the load Λ plays the role of the distinctive bifurcational parameter of the earlier formalism. Clearly the cubic coefficient $\mathscr{A}^{\mathrm{C}}_{111}$ is zero by the symmetry condition. Typically, $\mathscr{A}^{\mathrm{C}}_{1111}$ is positive for flat plates and negative for curved panels, so C is likely to be stable symmetric for stiffened plates and unstable symmetric for stiffened cylinders.

(*c*) *Interaction between local modes*

As the aspect (length-to-breadth) ratio of a flat plate is increased, there is a tendency for critical eigenvalues to bunch together (Timoshenko & Gere 1961). We are thus obliged to consider possible interactive effects between different local modes, since under the wavelength condition we are dealing with a long plate. We show that all cubic coefficients, and hence all contaminations of $\mathscr{A}_{ijkl}$ whatever the active coordinates, vanish for such interactions by use of a general symmetry condition. Again, the same is approximately true for stiffened cylinders, from the rapidly oscillating nature of the modes in question (Koiter 1976).

The usual condition for total symmetry, expressed for a perfect system with two modes (Supple 1973) as

$$A(u_1, u_2, \Lambda) = A(-u_1, u_2, \Lambda) = A(u_1, -u_2, \Lambda), \tag{48}$$

is too strong to be used in general here; it implies that symmetry in one mode is retained in the presence of the other, and breaks down, for example, if both modes have an odd number of half-waves longitudinally. We replace it with a weaker condition,

$$A(u_i, \Lambda) = A(-u_i, \Lambda), \tag{49}$$

which merely states that reversing all modes simultaneously gives identical energy levels (Golubitsky & Schaeffer 1979*b*, 1979*c*); this holds for all sinusoidal waveforms, by the same argument as before.

The cubic coefficients are now quickly eliminated. Allowing just two coincident modes at C, u_i and u_j, reversing them both and equating the cubic contributions, and remembering that $\mathscr{A}^{\mathrm{C}}_{iii} = 0$, we have,

$$A^{\mathrm{C}}_{iij}u_i^2u_j + A^{\mathrm{C}}_{ijj}u_iu_j^2 = 0, \tag{50}$$

with the tensor summation convention temporarily suspended.

We note that lack of contamination of cubics in equation (46) means that we can interchange freely between these A- and $\mathscr{A}$-derivatives. Equation (50) must hold for any ratio u_i/u_j; so $A^{\mathrm{C}}_{iij} = 0$. By a similar argument with three modes we now obtain

$$A^{\mathrm{C}}_{ijk} u_i u_j u_k, \tag{51}$$

summed over the three, which again must hold however we emerge from C in u_i–u_j–u_k-space. Thus, in general,

$$A^{\mathrm{C}}_{ijk} = 0. \tag{52}$$

Interactions on a quartic level can of course still arise between local modes (Supple 1973, Magnus & Poston 1977, Golubitsky & Schaeffer 1979*b*, *c*), but we comment no further on this. It is the absence of the cubics, which otherwise might contaminate the dominant interaction between local and overall modes, that is of interest here.

(*d*) *Overall mode*

For buckling in the overall mode alone, a stiffened system can usually be modelled by a corresponding, perfectly orthotropic material, thereby smearing out the effect of the stiffeners. Symmetry of the potential function in the mode, u_2, is then guaranteed by mirror-image reflexion as before, and we can write

$$\mathscr{A} = \tfrac{1}{24}\mathscr{A}^{\mathrm{C}}_{2222} u_2^4 + \tfrac{1}{2}\mathscr{A}'^{\mathrm{C}}_{22}(\Lambda - \Lambda^{\mathrm{C}})\, u_2^2 + \text{higher-order terms}, \tag{53}$$

for the perfect system. For a moderately wide stiffened plate this corresponds closely to the familiar buckling of the Euler strut, with a positive but very small value of $\mathscr{A}^{\mathrm{C}}_{2222}$. For a stiffened cylinder $\mathscr{A}^{\mathrm{C}}_{2222}$ is negative, and the critical state unstable.

Without the smearing approximation, a non-zero value of the cubic $\mathscr{A}^{\mathrm{C}}_{222}$ is sometimes found. Tvergaard (1973) for example develops such a term in his infinitely wide single-bay plate, resulting from anticlastic bending in the panels between the stiffeners; this lead to the hyperbolic umbilic model of interactive buckling (Hunt 1977), briefly discussed later. Koiter & Pignataro (1976) ignore transverse bending and direct stresses in their formulation, thereby guaranteeing $\mathscr{A}^{\mathrm{C}}_{222} = 0$. This must also be true for multi-bay panels with an even number of bays, and stiffened cylinders, by simple symmetry considerations as before.

(*e*) *Interaction between local and overall buckling*

For buckling in both modes simultaneously, the smearing approximation is invalid, and we lose its associated symmetries. Unless they are otherwise guaranteed (by symmetry in the cross section for example) the typical response is dominated by a $\mathscr{A}^{\mathrm{C}}_{112}$-term in the potential function; clearly this implies a loss of symmetry in the overall mode u_2. We continue to assume symmetry in u_1, even in the presence of u_2, the rapidly oscillating nature of the local mode justifying the assumption that it carries an even number of half-waves longitudinally. Thus, $\mathscr{A}^{\mathrm{C}}_{122} = 0$.

The essence of the interaction, and the appearance of $\mathscr{A}^{\mathrm{C}}_{112}$, can be captured with a simple model of the plate of figure 1. We follow Koiter & Pignataro (1976) in ignoring all transverse stresses and shear lag, and add the extra assumption that the neutral surface, where the load is applied, is inextensional. All of these effects could have been included, but only would have served to obscure the important interactive phenomenon.

Let us thus consider a typical section through the panel, as shown in figure 2. We define x- and y-axes which remain on the neutral surface, as in the inextensional strut formulation of Thompson & Hunt (1973). Deformation in the overall mode is resolved into an unspecified horizontal component and a vertical component $W(x)$, and similarly local deformation is resolved into an unspecified horizontal component and a vertical component $w(x, y)$ as shown. For our present purposes it is adequate to assume

$$w(x, y) = u_1 g(y)\, l \sin \pi x/l, \quad W(x) = u_2 L \sin \pi x/L, \tag{54}$$

$g(y)$ denoting the local mode profile in the transverse y-direction. We see that the overall mode exhibits no y-dependence.

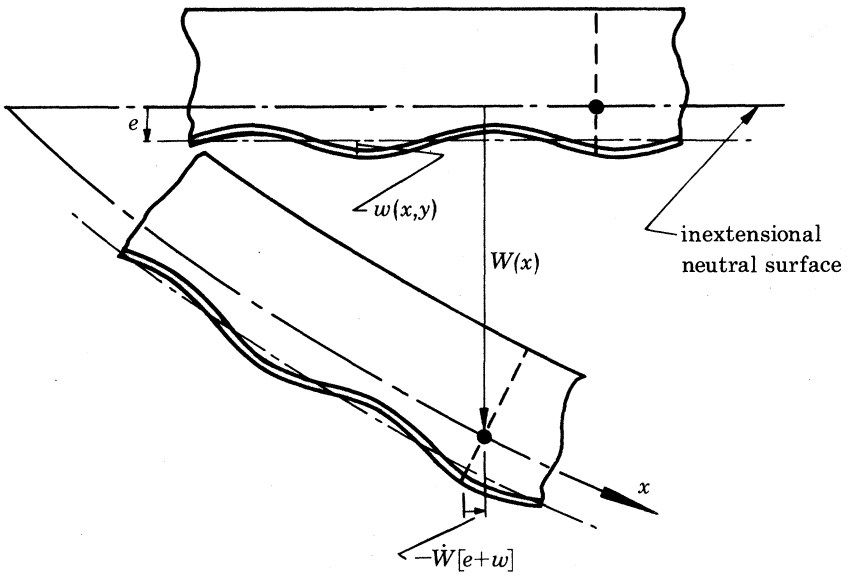

FIGURE 2. Simultaneous buckling in both local and overall modes of the stiffened plate, at a typical section y = constant.

We take first just local mode deformation, as shown at the top of the figure. Inextensibility of the neutral surface, and zero curvature, means that the panel must undergo longitudinal stretching. To a first order this change in length is

$$\delta L = \frac{1}{2}\int_0^L \dot{w}^2 \mathrm{d}x, \tag{55}$$

a dot denoting differentiation with respect to x. Assuming that it sets up a constant axial strain through the length, we have $\epsilon_x = \delta L/L$ in the deformed panel.

The stretching must involve some axial displacement, but we now assume that after this has taken place, during the overall deformation, plane sections remain plane and shear lag is ignored. Approximating for small angles, we thus have a further displacement in the x-direction of $-\dot{W}(e+w)$, e being the panel eccentricity, as shown. This sets up the additional axial strain

$$\epsilon_x = -\partial \dot{W}(e+w)/\partial x = -\ddot{W}(e+w) - \dot{W}\dot{w}. \tag{56}$$

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Substituting the assumed forms and performing the differentiations, we obtain the total axial strain in the panel as

$$\epsilon_x = \tfrac{1}{4}\pi^2 g^2 u_1^2 + \frac{e}{L}\pi^2 u_2 \sin\frac{\pi x}{L} + \pi^2 g u_1 u_2 \left(\frac{l}{L}\sin\frac{\pi x}{L}\sin\frac{\pi x}{l} - \cos\frac{\pi x}{L}\cos\frac{\pi x}{l}\right). \tag{57}$$

Having ignored all transverse stresses, we obtain the membrane energy of a particular panel by integrating $\frac{1}{2}E\epsilon_x^2$ over its volume, E being the Young modulus. The cross product of the first two terms gives

$$\mathscr{A}^{\mathrm{C}}_{112} = \pi^3 Ete \int_0^b g^2 \mathrm{d}y \tag{58}$$

for the panel, where t is its thickness and b its breadth. This is the cubic of interest.

It is interesting to observe that the deformation pattern of figure 2 is only admissible for vanishingly small displacements; the appearance of the transverse variation $g(y)$, and the assumption that plane sections remain plane, become essentially contradictory as deflexions grow in the combined mode. The problem remains with the introduction of neutral surface extensibility as a passive effect, as in the next section. It is precisely because, by equations (46), cubics are uncontaminated by passive coordinates that we obtain essentially the same result as Koiter & Pignataro's (1976) lower bound.

We note finally that, although this model is adequate for the dominant cubic, other contributions to the energy such as quartics and load-dependent terms are also of key analytical significance. However these can always be obtained by considering the two modes separately.

(*f*) *Neutral surface extensibility*

One feature of plate post-buckling which is missing from the above formulation, but which is clearly desirable for a good descriptive model, is the growth in significance of membrane effects as the panels deform into the non-developable (doubly curved) local mode; this accounts for the considerable post-buckling stiffness of a pure plate mode. We must thus introduce stretching of the neutral surface; this in turn eliminates the trivial nature of the fundamental path, and allows for another well known phenomenon of stiffened structures – the shift in the neutral surface away from the locally buckled panels as they lose effective stiffness. Such effects are not truly products of the interaction, and can be examined in the local mode alone. Again by equations (46) the coefficients thus obtained relate directly to their counterparts in the interactive buckling formulation.

Neutral surface extensibility is best treated as a passive effect. We thus suppose that, entirely as a result of local buckling, the neutral surface undergoes a uniform overall contraction across the width of an amount $u_3 L$; u_3 thus measures change from a uniformly contracted fundamental state, and is a genuine incremental coordinate. Assuming that this sets up a constant axial strain through the length, we have

$$\epsilon_x = \tfrac{1}{4}\pi^2 g^2 u_1^2 - u_3, \tag{59}$$

much as before (see equation (57)). Integrating again over the volume, we obtain the following coefficients of the potential function $A(u_1, u_3, \Lambda)$,

$$A^{\mathrm{C}}_{113} = -\tfrac{1}{2}\pi^2 ELt \int_0^b g^2 \mathrm{d}y, \quad A^{\mathrm{C}}_{33} = ELa, \quad A^{\mathrm{C}}_{1111} = \tfrac{3}{4}\pi^4 ELt \int_0^b g^4 \mathrm{d}y, \tag{60}$$

for a typical panel-stiffener combination of cross-sectional area a.

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Eliminating u_3 as a passive coordinate using the last of equations (46), we thus have

$$\begin{aligned}\mathscr{A}^{\mathrm{C}}_{1111} &= A_{1111} - 3(A_{113})^2/A_{33}|^{\mathrm{C}} \\ &= \tfrac{3}{4}\pi^4 ELt\left[\int_0^b g^4\,\mathrm{d}y - \frac{t}{a}\left(\int_0^b g^2\,\mathrm{d}y\right)^2\right],\end{aligned} \tag{61}$$

which agrees with Koiter & Pignataro (1976).

The coefficient remains unchanged with the further addition of overall mode u_2. The cross term A^{C}_{23}, which at first glance seems likely when the strain equations (57) and (59) are combined, vanishes on integration through the complete depth including stiffeners. The same does not apply to A^{C}_{113}, as with the earlier cubic, since $g(y)$ vanishes at the stiffeners.

(g) *Three potential-function models*

Taking all of the earlier discussion into account, we have three possible potential-function descriptions of the interaction and associated behaviour. Of these, it is clear that the parabolic umbilic catastrophe is the most important for stiffened structures, but we note that it has received less attention in the literature than the other two.

First we have the semi-symmetric points of bifurcation, given by

$$\mathscr{A} = \tfrac{1}{24}\mathscr{A}^{\mathrm{C}}_{1111}u_1^4 + \tfrac{1}{6}\mathscr{A}^{\mathrm{C}}_{222}u_2^3 + \tfrac{1}{2}\mathscr{A}^{\mathrm{C}}_{112}u_1^2u_2 + \tfrac{1}{2}(\Lambda - \Lambda^{\mathrm{C}})\,(\mathscr{A}'^{\mathrm{C}}_{11}u_1^2 + \mathscr{A}'^{\mathrm{C}}_{22}u_2^2) + \text{higher-order terms.} \tag{62}$$

This corresponds to either the hyperbolic or the elliptic umbilic catastrophe (Hunt 1977), although in the canonical form the $\mathscr{A}^{\mathrm{C}}_{1111}$-term is omitted. Tvergaard's (1973) stiffened plate exhibits this, but its range of validity is very limited (Koiter 1976).

Secondly, replacing $\mathscr{A}^{\mathrm{C}}_{222}$ with $\mathscr{A}^{\mathrm{C}}_{2222}$ as discussed earlier, we have the paraclinal point of bifurcation

$$\mathscr{A} = \tfrac{1}{24}\mathscr{A}^{\mathrm{C}}_{1111}u_1^4 + \tfrac{1}{24}\mathscr{A}^{\mathrm{C}}_{2222}u_2^4 + \tfrac{1}{2}\mathscr{A}^{\mathrm{C}}_{112}u_1^2u_2 + \tfrac{1}{2}(\Lambda - \Lambda^{\mathrm{C}})\,(\mathscr{A}'^{\mathrm{C}}_{11}u_1^2 + \mathscr{A}'^{\mathrm{C}}_{22}u_2^2) + \text{higher-order terms.} \tag{63}$$

This corresponds to the parabolic umbilic catastrophe, although $\mathscr{A}^{\mathrm{C}}_{1111}$ is again omitted from the canonical form.

Finally, we have the fully symmetric point of bifurcation, which must apply, for example, when $e = 0$;

$$\mathscr{A} = \tfrac{1}{24}\mathscr{A}^{\mathrm{C}}_{1111}u_1^4 + \tfrac{1}{4}\mathscr{A}^{\mathrm{C}}_{1122}u_1^2u_2^2 + \tfrac{1}{24}\mathscr{A}^{\mathrm{C}}_{2222}u_2^4 + \tfrac{1}{2}(\Lambda - \Lambda^{\mathrm{C}})\,(\mathscr{A}'^{\mathrm{C}}_{11}u_1^2 + \mathscr{A}'^{\mathrm{C}}_{22}u_2^2) + \text{higher-order terms.} \tag{64}$$

This corresponds to the double-cusp catastrophe.

All of these refer to the perfect system at complete coincidence. The key interaction term of the first two is $\mathscr{A}^{\mathrm{C}}_{112}$ as we have seen. In the third this is replaced by $\mathscr{A}^{\mathrm{C}}_{1122}$, which is liable to contamination from higher local modes. Equations (46) give,

$$\mathscr{A}^{\mathrm{C}}_{1122} = A^{\mathrm{C}}_{1122} - \sum_{\alpha=3}^{n}\frac{1}{A_{\alpha\alpha}}\left[A_{\alpha 11}A_{\alpha 22} + 2(A_{\alpha 12})^2\right]\Big|^{\mathrm{C}}. \tag{65}$$

This may well exhibit a collection of non-zero $A^{\mathrm{C}}_{\alpha 12}$-contributions, and localized analysis is thus considerably more complicated.

(h) Critical point separation giving a butterfly catastrophe

We now briefly consider the separation of the two contributing bifurcations, which can be achieved analytically with the introduction of a splitting parameter σ as in the following semi-symmetric analysis. We limit the discussion to the first two potential functions of the previous section, with non-zero $\mathscr{A}^{\mathrm{C}}_{112}$.

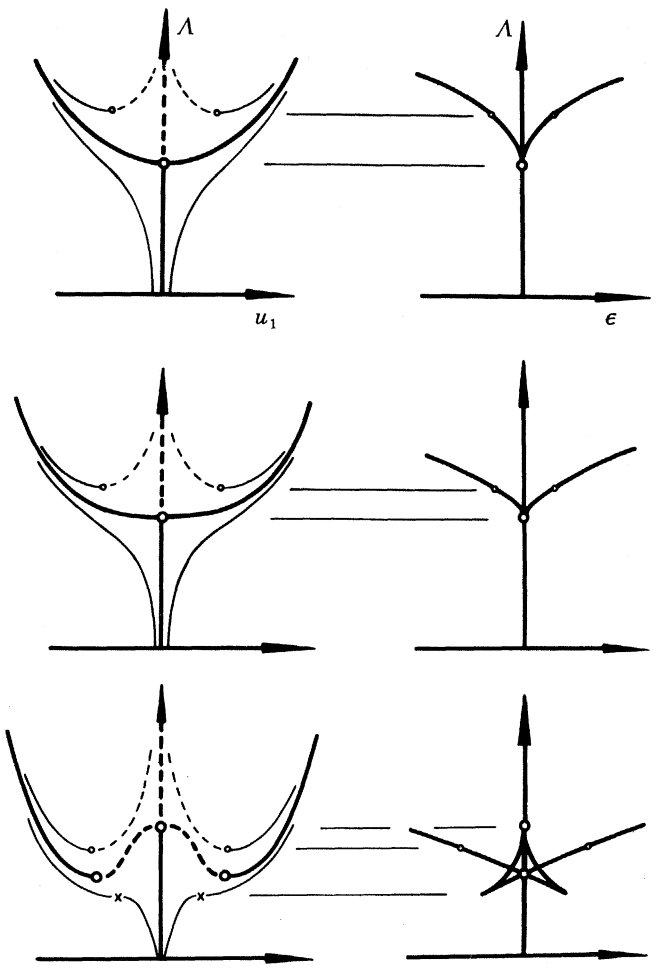

FIGURE 3. Equilibrium paths and imperfection sensitivity about a butterfly catastrophe, with $\mathscr{A}^{\mathrm{C}}_{111111}$ positive. For the stiffened plate, with the two bifurcations well separated, the post-buckling is fully stable (top). As they converge, we meet the butterfly point (centre), and buckling in the local mode becomes unstable (bottom).

Let us suppose that local buckling occurs first with increasing load, and $\mathscr{A}^{\mathrm{C}}_{1111}$ is positive ($\mathscr{A}$ defined as for $m = 2$). Treating this as a distinct bifurcation in u_1, we now eliminate the overall mode u_2 as a passive coordinate, dropping to $m = 1$. Equations (46) give

$$\mathscr{A}^{\mathrm{C}}_{1111} = A^{\mathrm{C}}_{1111} - 3(A_{112})^2/A_{22}|^{\mathrm{C}}, \tag{66}$$

where the coefficients of the right-hand side are of course the same as the $\mathscr{A}$-derivatives discussed earlier. We note that A^{C}_{22} is positive (Thompson & Hunt 1973), but approaches zero as the two bifurcations coalesce; there must thus be some finite separation at which $\mathscr{A}^{\mathrm{C}}_{1111} = 0$,

the passive contamination being just enough to eliminate the positive A^{C}_{1111}. This is a butterfly catastrophe point.

Stability at the butterfly point is controlled by a sixth-order term, which is beyond the scope of this analysis. However, taking it as positive, we would expect the equilibrium path behaviour and imperfection-sensitivity shown schematically in figure 3. We note that, although buckling is initiated in mode u_1, the contamination implies that it quickly develops into the combined mode; this effect is not shown in figure 3.

The phenomenon is clearly of some design significance. It defines the limit of purely stable structural response, and as such is given some analytical prominence by Koiter (1976). It is also closely related to the so-called 'reduced Euler load' (Thompson *et al.* 1976). We note that it is certainly amenable to general analysis, although we explore it no further here.

(*i*) *Koiter's amplitude modulation of the local mode*

Finally, we briefly discuss one further refinement which can be brought to the model. In all of Koiter's studies of the interaction between local and overall buckling, including stiffened structures, it is assumed that the local mode amplitude is not constant along the length, but is a 'slowly-varying' function of x; thus the local mode is modulated in the presence of the overall.

The analytical implications of this are quite considerable. For example, to determine from first principles the amplitude function giving the minimum energy state requires an application of the calculus of variations. It is clearly important here whether local or overall buckling arises first. Nevertheless, appropriate functions for each situation are given explicitly by Koiter (1976), and these could be used directly to arrive at a modified two-degree-of-freedom model. We explore this no further at present.

The refinement is of considerable practical significance, allowing local buckling to be a truly localized effect. In multi-bay plates, for example, it sets the amplitude to zero when local and overall buckling act against each other (as with overall deformation away from the stiffeners in figure 1). This introduces an $\mathscr{A}^{C}_{112}$-term into such systems which otherwise, with an even number of half-waves, carry a seemingly enforced symmetry in u_2. Entirely similar conclusions can also be drawn for stiffened cylinders. We finally note that such amplitude modulation clearly cannot be simply described from a modal (Fourier-type) formulation.

4. Semi-symmetric bifurcation

(*a*) *Comprehensive analysis*

Our treatment of stiffened structures leans largely on the elimination-of-passive-coordinates routine, no analysis of the bifurcation itself being attempted. We close with a full quantitative description of the simplest of the three proposed potential function models, the semi-symmetric point of bifurcation. We include four controls, the load, two imperfections, and a splitting parameter. We note that the last of these is not necessary for a universal unfolding in the sense of Thom (1975), but is of considerable practical importance.

We thus write potential function (62), previously given for the perfect system only, to include all of these parameters:

$$\mathscr{A} = \tfrac{1}{6}\mathscr{A}^{C}_{222}u_2^3 + \tfrac{1}{2}\mathscr{A}^{C}_{112}u_1^2u_2 + \tfrac{1}{2}(\Lambda-\Lambda^{C})(\mathscr{A}'^{C}_{11}u_1^2 + \mathscr{A}'^{C}_{22}u_2^2) + \tfrac{1}{2}\mathscr{A}'^{C}_{11}\sigma u_1^2 + \mathscr{A}^{1C}_{1}\epsilon^1u_1 + \mathscr{A}^{2C}_{2}\epsilon^2u_2 + \text{higher-order terms.} \quad (67)$$

Here the $\mathscr{A}^{\mathrm{C}}_{1111}$-term is omitted, as in the canonical form (Thom 1975), and we must insist that $\mathscr{A}^{\mathrm{C}}_{222}$ and $\mathscr{A}^{\mathrm{C}}_{112}$ are both non-zero. However we can and do set $\mathscr{A}^{1\mathrm{C}}_{2}$ and $\mathscr{A}^{2\mathrm{C}}_{1}$ to zero without loss of generality, so ϵ^1 and ϵ^2 are *principal imperfections*, related directly to u_1 and u_2 respectively. We see that the splitting parameter σ, when introduced in this way, allows for a direct measure of critical load separation on the fundamental path.

The lowest-order equilibrium perturbation equations (29) of interest become

$$\left.\begin{aligned} 2\mathscr{A}_{112}u_1^{(1)}u_2^{(1)} + 2\mathscr{A}'_{11}u_1^{(1)}\Lambda^{(1)} + 2\mathscr{A}'_{11}u_1^{(1)}\sigma^{(1)} + \mathscr{A}^{1}_{1}\epsilon^{1(2)}\big|^{\mathrm{C}} &= 0,\\ \mathscr{A}_{112}(u_1^{(1)})^2 + \mathscr{A}_{222}(u_2^{(1)})^2 + 2\mathscr{A}'_{22}u_2^{(1)}\Lambda^{(1)} + \mathscr{A}^{2}_{2}\epsilon^{2(2)}\big|^{\mathrm{C}} &= 0, \end{aligned}\right\} \qquad (\mathbf{E})$$

and the critical state equations (31) are

$$\left.\begin{aligned} \mathscr{A}_{112}x_2u_1^{(1)} + \mathscr{A}_{112}x_1u_2^{(1)} + \mathscr{A}'_{11}x_1\Lambda^{(1)} + \mathscr{A}'_{11}x_1\sigma^{(1)}\big|^{\mathrm{C}} &= 0,\\ \mathscr{A}_{112}x_1u_1^{(1)} + \mathscr{A}_{222}x_2u_2^{(1)} + \mathscr{A}'_{22}x_2\Lambda^{(1)}\big|^{\mathrm{C}} &= 0, \end{aligned}\right\} \qquad (\mathbf{C})$$

which take the special form of equations (33) for m-fold criticality,

$$\left.\begin{aligned} \mathscr{A}_{112}u_2^{(1)} + \mathscr{A}'_{11}\Lambda^{(1)} + \mathscr{A}'_{11}\sigma^{(1)}\big|^{\mathrm{C}} &= 0,\\ \mathscr{A}_{112}u_1^{(1)}\big|^{\mathrm{C}} &= 0,\\ \mathscr{A}_{222}u_2^{(1)} + \mathscr{A}'_{22}\Lambda^{(1)}\big|^{\mathrm{C}} &= 0. \end{aligned}\right\} \qquad (\boldsymbol{m}\mathbf{C})$$

Finally the extra equation (35), specifying secondary bifurcations as opposed to limit points, becomes

$$\mathscr{A}'_{11}x_1u_1^{(1)} + \mathscr{A}'_{22}x_2u_2^{(1)}\big|^{\mathrm{C}} = 0. \qquad (\mathbf{B})$$

Just which of these equations are to be used, which terms if any are absent, and how they are to be solved depend precisely on the problem in hand. For example, if we are to consider the post-buckling of the perfect system, $\epsilon^1 = \epsilon^2 = 0$, we would use equations (E) with

TABLE 1. FIRST-ORDER ANALYSIS OF SEMI-SYMMETRIC BIFURCATION

(The appropriate equations for each situation are as shown, subject to the conditions arising from the exclusion of certain control parameters.)

	equations	conditions	solution	
post-buckling of the perfect system	**E**	$\sigma^{(1)} = 0$, $\epsilon^{1(2)} = \epsilon^{2(2)} = 0$	direct	complete coincidence $\sigma = 0$
imperfection sensitivity on the symmetric section	**E, C**	$\sigma^{(1)} = \epsilon^{1(2)} = 0$	direct	complete coincidence $\sigma = 0$
full imperfection sensitivity	**E, C**	$\sigma^{(1)} = 0$	generalized imperfection	complete coincidence $\sigma = 0$
secondary bifurcations	**E, C, B**	$\sigma^{(1)} = 0$	direct	complete coincidence $\sigma = 0$
post-buckling of the perfect system	**E**	$\epsilon^{1(2)} = \epsilon^{2(2)} = 0$	direct	near coincidence
bifurcations of the perfect system	**E, C**	$\epsilon^{1(2)} = \epsilon^{2(2)} = 0$	direct	near coincidence
imperfection sensitivity on the symmetric section	**E, C**	$\epsilon^{1(2)} = 0$	direct	near coincidence
locus of m-fold critical point (umbilic point)	**E, *m*C**		direct	near coincidence
full imperfection sensitivity	**E, C**		generalized imperfection, generalized loading parameter	near coincidence
secondary bifurcations	**E, C, B**		generalized loading parameter	near coincidence

$\epsilon^{1(2)} = \epsilon^{2(2)} = 0$, and with $\sigma^{(1)} = 0$ if we are to be concerned just with complete coincidence. The significant alternatives are presented in table 1.

The generalized imperfection and generalized loading parameter can be used as aids to the solution of the equations when appropriate, and further simplifications which can also sometimes be exploited include specification of the perturbation parameter s and normalization of the local eigenvector x_j. As an example of the former, it is sometimes useful in imperfection-sensitivity studies to specify s as the positive square root of generalized imperfection, written $\epsilon^{\frac{1}{2}+}$ (Hunt 1977, 1979). Writing ϵ as a Taylor expansion about C we thus have

$$\epsilon = \epsilon^{(1)C}s + \tfrac{1}{2}\epsilon^{(2)C}s^2 + \tfrac{1}{6}\epsilon^{(3)C}s^3 + \ldots, \tag{68}$$

and equating coefficients,

$$\epsilon^{(1)C} = \epsilon^{(3)C} = \epsilon^{(4)C} = \ldots = 0, \tag{69}$$

$$\epsilon^{(2)C} = 2.$$

Sometimes other interpretations of s may be useful, but we must ensure that s is capable of describing all possible solutions; for this reason it may be left undefined and solutions obtained

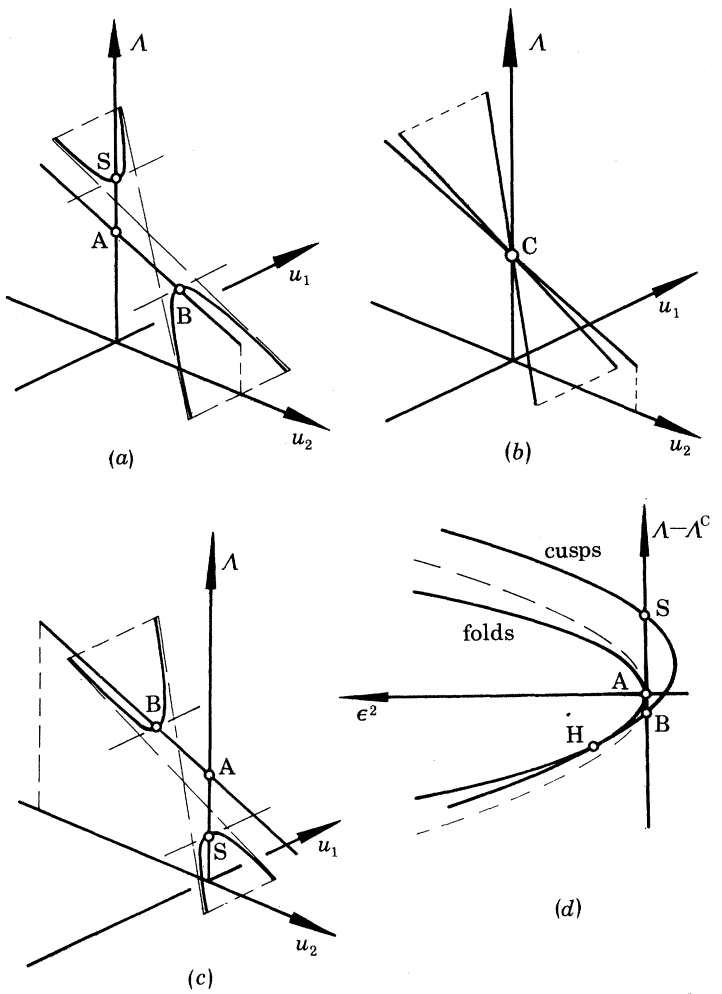

FIGURE 4. The homoeoclinical point of bifurcation: (*a*)–(*c*) show patterns of equilibria for a perfect system with $\sigma < 0$, $\sigma = 0$, and $\sigma > 0$ respectively; (*d*) shows the failure locus (imperfection sensitivity) with the addition of a symmetry-preserving imperfection ϵ^2.

in a rate space $u_i^{(1)} - \Lambda^{(1)}$ say, although these become the path tangents when mapped directly into the corresponding coordinate space $u_i - \Lambda$ (Thompson & Hunt 1973). Normalization of the local eigenvector involves simply setting one of the x_i to unity; thus $x_1 = 1$ takes care of all possibilities except $x_1 = 0$, which must then be explored as a special case (Hunt 1977).

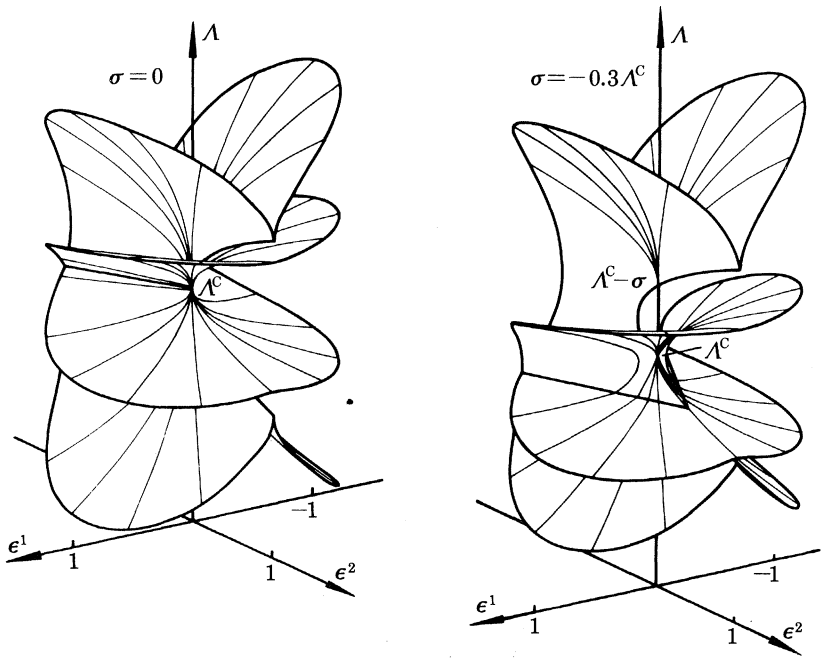

FIGURE 5. The full imperfection sensitivity of the homoeoclinical point of bifurcation for the Tvergaard stiffened plate, at complete coincidence and with the bifurcations separated. Both exhibit the unique topology of the hyperbolic umbilic catastrophe.

(*b*) *Subclassification of semi-symmetric branching*

Semi-symmetric branching has three separate bifurcational manifestations, the difference depending on the number of post-buckling paths of the perfect system at complete coincidence and the form of the imperfection sensitivity; we refer to these as monoclinal, anticlinal and homoeoclinal branching. Full imperfection-sensitivity surfaces at complete and near coincidence for all three are computed and plotted by Hunt *et al.* (1979), each being related directly to a different starting condition of the propped cantilever model due to Thompson & Gaspar (1977). Here we shall only illustrate the general analysis of one of the three, the homoeoclinal point of bifurcation.

We shall not give all the results of table 1 since they have, in the main, been generated by isolated perturbation schemes in the past; the treatment of full coincidence can be found in Hunt (1977), and of near coincidence in Hunt (1979). However, we do show the main results, in the behaviour of the equilibrium paths of the perfect system (figure 4*a–c*), the imperfection sensitivity on the symmetric section (figure 4*d*), and the full imperfection-sensitivity (figure 5). The latter has been specifically plotted for the Tvergaard (1973) plate, and has also been related to the propped cantilever as discussed above.

(c) *Numerical search procedure*

This section might be omitted by all readers except those with an interest in the details of the imperfection-sensitivity analysis.

A polar representation of controls may be entirely appropriate for imperfections, when we come to plot the final results, but not necessarily for the loads; it is of more interest to view the imperfection sensitivity, for instance, at a constant value of splitting parameter σ, than a constant angular polar coordinate ϕ. We could translate the second view into the first by fixing the load state and using ϵ as a dependent variable, but this is contrary to the nature of imperfection-sensitivity plots, which are invariably visualized as imperfection independent and load dependent. For a systematic plotting procedure we would clearly like to stick with convention, but since σ is previously specified we are now obliged to introduce a numerical search procedure, necessarily performed on a computer.

Let us thus consider the plotting routine for a typical imperfection-sensitivity surface with the four controls of Λ, σ, ϵ^1 and ϵ^2, as shown in figure 5. We first fix the values of ϵ^i, giving θ and ϵ of the polar representation. Applying transformation (37), we can write down the equations for the imperfection ray, and considering for the moment $\sigma = 0$, they can be solved for the set of $\Lambda^{(1)C}$; hence, knowing ϵ, we obtain a first-order estimate of critical loads.

But when σ is non-zero, the situation is not so simple; the generalized load approach can be used, but we cannot fix the correct load ray without knowing Λ and it is this that we are seeking in the analysis. We overcome the problem by scanning the complete load space incrementally, using a numerical search involving a reduction in increment size, to home in on situations where a solution to the equations coincides with the value obtained from the polar transformation. The process of course implies greatly increased computer time, and it may therefore be advisable in problems of near coincidence to force the bifurcations together by introducing a splitting parameter. However, the analyst must be aware of the topological significance of the manoeuvre, in the light of the criteria of structural stability and universal unfolding.

Concluding remarks

The paper supplies a carefully worked algorithm for the analysis of compound bifurcation, in a form suitable for the application of numerical procedures; it could also be applied to distinct bifurcations, but techniques already abound for their study (see, for example, Thompson & Hunt 1973). Of course, the intricacies of the analysis must increase with the complexity of the key underlying phenomenon, and thus for some problems of compound bifurcation the paper can only indicate the way forward. Here the simplifying concepts such as generalized imperfection and generalized load will remain useful, but may need some rethinking.

The stiffened-structure formulation raises some points of general interest. The first lies in the use of symmetry conditions and the elimination of passive coordinates. In combination, these are seen to provide a simple tool for assessing the nature of instabilities, and could be extended to other systems. They do not, of course, tell us whether a symmetric bifurcation is stable or unstable, for which we need to quantify certain coefficients. Nevertheless, the approach seems useful, particularly for problems of compound bifurcation.

Secondly, the treatment draws attention to the ways in which the catastrophes of Thom (1975) nest inside one another. If the substrata of a phenomenon are known completely, light

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is shed on the ways in which a perturbed version can appear, and remedies can be sought for the problems of all-too-local effects. We see this in the Tvergaard (1973) stiffened plate, where in the overall mode alone an asymmetric point of bifurcation (fold) is better seen as a perturbed symmetric bifurcation (cusp); less trivial, in the interactive case the hyperbolic umbilic of figure 5 appears in the substrata of a parabolic umbilic, which in turn might be seen, in the large, as part of a perturbed double cusp.

In conclusion, we feel that competent analysis requires a combination of qualitative understanding and numerical expertise.

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